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Fibrations on cyclic coverings of the projective plane

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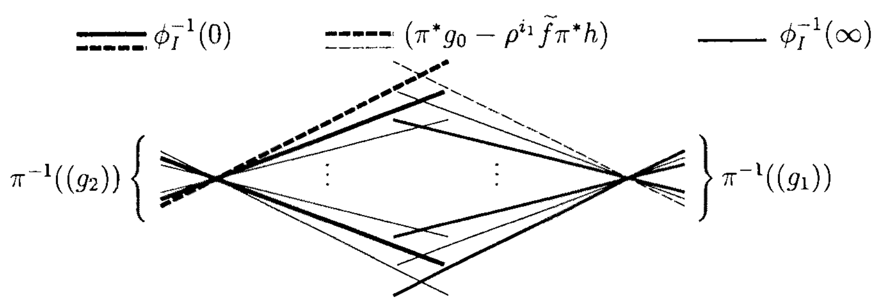
Introduction

Let G be a finite group. We call a Galois covering whose covering transformation group is isomorphic to G, a G-covering. Let r and n be integers greater than 1. Let $\lambda : Y \to \mathbf{P}^2$ be a $\mathbf{Z}_r$-covering of the projective plane ramifying along a curve (f) of degree n. In this paper, we study how many fibrations Y has and what singular fibers appear on those fibrations. Here we call a holomorphic map from Y onto a curve whose fibers are connected, a fibration on Y. We note that if (f) is irreducible, then n is a multiple of r and Y is biholomorphic to $X_{f,r}$, because $H_1(\mathbf{P}^2 \setminus (f), \mathbf{Z}) = \mathbf{Z}_n$ (see Proposition 1.3 in Chapter 4 of [1]). Here $X_{f,r}$ is the subvariety defined by $\xi^r - f(z_0, z_1, z_2) = 0$ in the total space of a line bundle over $\mathbf{P}^2$ of degree $s := n/r$. In addition if $n = r$, then Y is biholomorphic to the hypersurface of $\mathbf{P}^3$ defined by $z_3^r - f(z_0, z_1, z_2) = 0$. First, we give an example of fibrations on $X_{f,r}$. Assume that n is a multiple of r, let $X = X_{f,r}$ and let $\pi : X \to \mathbf{P}^2$ be the projection. Then π is a $\mathbf{Z}_r$-covering and there exist an element $\widetilde{f}$ of $H^0(X, \pi^*\mathcal{O}(s))$ and a generator σ of $\mathrm{Gal}(X/\mathbf{P}^2)$ satisfying $\widetilde{f}^r = \pi^* f$ and $\sigma^* \widetilde{f} = \rho \widetilde{f}$, where $\rho = \exp(2\pi\sqrt{-1}/r)$. Let m, p and q be positive integers satisfying $m \geq s$ and $p + q = r$. Assume that there exist homogenous polynomials g_0, g_1, g_2 and h of degree m, mp, mq and $m - s$, respectively, of z_0, z_1, z_2 satisfying

$$fh^r = g_0^r + g_1 g_2 \quad \text{and} \quad (g_0) \cap (g_1) \cap (g_2) = \emptyset.$$

For a subset $I = \{i_1, i_2, \ldots, i_p\}$ of $\{1, 2, \ldots, r\}$ with $|I| = p$, we define a meromorphic map $\phi_I : X \to \mathbf{P}^1$ onto $\mathbf{P}^1$ by $((\pi^* g_0 - \rho^{i_1} \widetilde{f} \pi^* h)(\pi^* g_0 - \rho^{i_2} \widetilde{f} \pi^* h) \cdots (\pi^* g_0 - \rho^{i_p} \widetilde{f} \pi^* h), \pi^* g_1)$. In Section 1, we prove the following.

Theorem 1. *If g_1 and g_2 are reduced, then ϕ_I is a holomorphic map.*



The above theorem shows that if (f) is the Fermat curve $(z_0^n + z_1^n + z_2^n)$, then X has $3\left(\binom{n}{s}\binom{r}{1} + \binom{n}{2s}\binom{r}{2} + \cdots + \binom{n}{n/2-s}\binom{r}{r/2-1} + \binom{n}{n/2}\binom{r}{r/2}/2\right)$ or $3\left(\binom{n}{s}\binom{r}{1} + \binom{n}{2s}\binom{r}{2} + \cdots + \binom{n}{(n-s)/2}\binom{r}{(r-1)/2}\right)$ fibrations, accordingly as r is even or odd, because there exist $\binom{n}{sp}$ pairs of polynomials g_1 and g_2 of degree sp and $n - sp$, respectively, satisfying $g_1 g_2 = z_i^n + z_j^n$. On the other hand, if $\phi_I^{-1}(0)$ or $\phi_I^{-1}(\infty)$ is connected, then ϕ_I is a fibration, i.e., all fibers $\phi_I^{-1}(t)$ are connected, and the genus of generic fibers of ϕ_I is equal to $mpq(n - n/r - 3)/2 + 1$ (Proposition 7). In particular, when h is a constant and $p = 1$, it is equal to $n(r-1)(n - n/r - 3)/(2r) + 1$. Conversely, we show at the end of Section 1 that when $n - n/r > 3$, any fibration $\phi : X \to \mathbf{P}^1$ on X the genus of whose generic fibers is equal to $n(r-1)(n - n/r - 3)/(2r) + 1$, coincides with one of the holomorphic maps $\phi_{\{i\}}$ in Theorem 1 for certain polynomials g_0, g_1 and g_2 of degree n/r, n/r and $n(r-1)/r$, respectively, satisfying $f = g_0^r + g_1 g_2$.

We can see the structure of $\phi_I^{-1}(0)$ and $\phi_I^{-1}(\infty)$, which consist of parts of the irreducible components of $\pi^{-1}((g_2))$ and $\pi^{-1}((g_1))$, respectively. For example, if $h = 1$, $g_0 = z_0$ and $g_1 = z_1$ $(f = z_0^r + z_1 g_2)$, then $\phi_{\{i\}}^{-1}(0)$ is biholomorphic to the curve (g_2) of degree $r - 1$ and $\phi_{\{i\}}^{-1}(\infty)$ consists of $r - 1$ rational curves meeting at a point. On the other hand, if r is even, $h = z_0$, $g_0 = z_1 z_2$, $g_1 = -z_0^r - z_1^r$ and $g_2 = z_0^r + z_2^r$ $(-f = z_0^r + z_1^r + z_2^r)$, then $\phi_I^{-1}(0)$ and $\phi_I^{-1}(\infty)$ consist of $r^2/2$ rational curves. In particular, when $r = 4$, ϕ_I is an elliptic fibration and $\phi_{\{1,2\}}^{-1}(0)$, $\phi_{\{1,2\}}^{-1}(\infty)$ are singular fibers of type I_8. Moreover, when h is a constant, $p = 1$ and (f) is non-singular, $\phi_{\{i\}}^{-1}(t)$ are biholomorphic to the curves $(((g_0 - tg_1)^r - (g_0^r + g_1 g_2))/g_1)$ for $t \in \mathbf{C}$. Hence in this case we can see what singular fibers appear on the fibrations $\phi_{\{i\}}$, besides $\phi_{\{i\}}^{-1}(0)$ and $\phi_{\{i\}}^{-1}(\infty)$.

In the case that $m = 1$ and $n = r = 2$, (f) is a smooth conic and the images $(g_2 + 2tg_0 - t^2 g_1)$ under π of the fibers $\phi_{\{1\}}^{-1}(t)$ of $\phi_{\{1\}}$ are tangents of (f). Moreover, then X is biholomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Let $\mu : Z \to \mathbf{P}^2$ be a holomorphic map from a compact analytic space Z onto $\mathbf{P}^2$ and let $(\mu^* f) = \sum_{i=1}^{l} c_i D_i$. Then there exists a double covering $Y \to Z$ ramifying along $\sum_{c_i \notin 2\mathbf{Z}} D_i$. and μ induces a surjective holomorphic map from Y to $X \simeq \mathbf{P}^1 \times \mathbf{P}^1$. Hence Y has fibrations. Conversely, we have:

Theorem 2. *Let $\lambda : Y \to Z$ be a double covering of a compact analytic space Z. Assume that there exist a surjective holomorphic map $\phi : Y \to B$ from Y to a curve B and no surjective holomorphic maps from Z to curves. Then there exists a surjective holomorphic map $\mu : Z \to \mathbf{P}^2$ satisfying $\mu(\Delta_\lambda) = (z_0^2 + z_1 z_2)$ and the following commutative diagram holds, where Δ_λ is the branch locus of λ and p is the projection.*

$$
\begin{array}{ccccccc}
B & \longrightarrow & \mathbf{P}^1 & & & & (\xi_0, \xi_1) \\
\phi \uparrow & & \uparrow p & & & & \uparrow \\
Y & \longrightarrow & \mathbf{P}^1 \times \mathbf{P}^1 & \supset & (diagonal) & & ((\xi_0, \xi_1), (\eta_0, \eta_1)) \\
\lambda \downarrow & & \downarrow \pi & & \downarrow & & \downarrow \\
Z & \xrightarrow{\mu} & \mathbf{P}^2 & \supset & (z_0^2 + z_1 z_2) & & (\xi_0\eta_1 + \xi_1\eta_0, -2\xi_0\eta_0, 2\xi_1\eta_1)
\end{array}
$$

In Section 2, we prove the above theorem. The images under the composite $\mu \circ \lambda$ of the connected components of the fibers of ϕ are tangents of the conic $(z_0^2 + z_1 z_2)$. Hence the pull-backs under μ of generic tangents of $(z_0^2+z_1z_2)$ are irreducible, if and only if the fibers of the composite $Y \xrightarrow{\phi} B \to \mathbf{P}^1$ are connected, i.e., ϕ is a fibration and $B \simeq \mathbf{P}^1$.

Corollary 3. *Let $\lambda : Y \to \mathbf{P}^N$ be a double covering of the projective space of dimension N greater than 2. Then there exist no surjective holomorphic maps from Y to curves.*

Corollary 4. *Let n be an even number and let $\lambda : Y \to \mathbf{P}^2$ be a double covering of the projective plane ramifying along a curve (f) of degree n. If Y has a fibration, then there exist polynomials g_0, g_1, g_2 of degree m and h of degree $m-n/2$ satisfying $fh^2 = g_0^2+g_1g_2$ and $(g_0)\cap(g_1)\cap(g_2)=\emptyset$.*

If $m=n/2$, i.e., h is a constant, then the holomorphic map $\mu : \mathbf{P}^2 \to \mathbf{P}^2$ defined by (g_0, g_1, g_2) does not ramify along the conic $(z_0^2+z_1z_2)$ and hence the pull-backs L under μ of generic tangents of $(z_0^2+z_1z_2)$ are non-singular. On the other hand, if $m>n/2$, then μ ramifies along $(z_0^2+z_1z_2)$ and hence L have singularities. When (f) and (h) have no common irreducible components, and (h) is reduced, L have $(m-n/2)m/2$ nodes, because $\deg(\mu)=m^2$ and L are tangent to (f) at $mn/2$ points. For example, if $g_0 = z_0^4+z_0^2z_1z_2+z_1^2z_2^2$, $g_1=-z_1^4$ and $g_2=z_2^4$, then $g_0^2+g_1g_2 = z_0^2(z_0^6+2z_0^4z_1z_2+3z_0^2z_1^2z_2^2+2z_1^3z_2^3)$ and the pull-back $(2z_0^4+2z_0^2z_1z_2+2z_1^2z_2^2-cz_1^4-c^{-1}z_2^4)$ of a generic tangent $(2z_0+cz_1-c^{-1}z_2)$ of the conic $(z_0^2+z_1z_2)$ under the holomorphic map defined by (g_0, g_1, g_2), is an irreducible quartic curve with two nodes at $(0, 1, \pm c^{1/2})$.

In Section 3, applying the above corollary, we calculate the number of the fibrations on a double covering Y of $\mathbf{P}^2$ ramifying along a quartic curve (f) with only nodes as singularities. When $n=4$, for any fibration on Y, the holomorphic map μ in Theorem 2 is obtained by quadratics g_0, g_1 and g_2 satisfying $f=g_0^2+g_1g_2$, i.e., the pull-backs of tangents of $(z_0^2+z_1z_2)$ under μ are quadrics (Proposition 12). In particular, when f is the product of two quadratics, we can write those quadratics g_0, g_1 and g_2 explicitly. On the other hand, if the curve (f) has two bitangents L_1 and L_2, then Y has a fibration with a fiber F satisfying $[Y \to \mathbf{P}^2](F) = L_1+L_2$ (Proposition 13), i.e., there exist quadratics g_0, g_1 and g_2 satisfying $f=g_0^2+g_1g_2$ and $(g_1)=L_1+L_2$. Moreover, any fibration on Y has at most six fibers whose images under the covering map $Y \to \mathbf{P}^2$ consist of two bitangents of (f). Hence Y has many fibrations for a generic quartic curve (f). Therefore, $\mathbf{Z}_4$-coverings of $\mathbf{P}^2$ ramifying along (f) also have many fibrations induced from those on Y, besides those obtained by Theorem 1. In Section 4, we examine what singular fibers appear on those fibrations. For example, if $f = z_0^2z_1^2+z_1^2z_2^2+z_2^2z_0^2$, then the minimal resolution of Y has an elliptic fibration with three singular fibers of type I_2^*, which is induced from the holomorphic map $(z_0^2, -z_0^2-z_1^2, z_0^2+z_2^2) : \mathbf{P}^2 \to \mathbf{P}^2$ (see Examples 3 and 6).

1 Proof of Theorem 1

We continue to use the notation in Introduction. Let C and D be the curves on $\mathbf{P}^2$ defined by $g_2=0$ and $g_1=0$, respectively.

Lemma 5. *For each irreducible component C' (resp. D') of C (resp. D), the number of the irreducible components of $\pi^{-1}(C')$ (resp. $\pi^{-1}(D')$) is equal to r.*

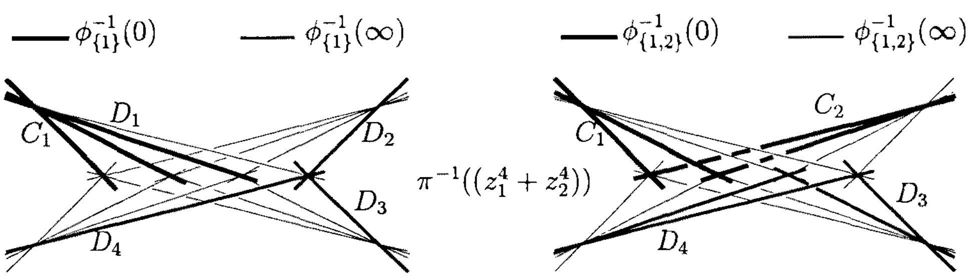
Proof. Note that $\left(\pi^*g_0 - \tilde{f}\pi^*h\right)\left(\pi^*g_0 - \rho\tilde{f}\pi^*h\right)\cdots\left(\pi^*g_0 - \rho^{r-1}\tilde{f}\pi^*h\right) = -\pi^*(g_1g_2)$ ($\in H^0(X, \pi^*\mathcal{O}(mr))$) has 0 of order 1 along $\pi^{-1}(C')$. Let C_0' be an irreducible component of $\pi^{-1}(C')$. Then there exists an integer j such that $(\pi^*g_0 - \rho^j\tilde{f}\pi^*h)$ has 0 along C_0'. Hence $(\sigma^i)^*(\pi^*g_0 - \rho^j\tilde{f}\pi^*h) = \pi^*g_0 - \rho^{j+i}\tilde{f}\pi^*h$ has 0 along $\sigma^{-i}C_0'$ for each i. Therefore, $\sigma^{-i}C_0' \neq C_0'$, if $0 < i < r$. □

Let C_i (resp. D_i) be the sum of the irreducible components of $\pi^{-1}(C)$ (resp. $\pi^{-1}(D)$) contained in $(\pi^*g_0 - \rho^i\tilde{f}\pi^*h)$. Then $\sigma^{i-1}(C_i) = C_1$ (resp. $\sigma^{i-1}(D_i) = D_1$) and $\pi^{-1}(C) = C_1 + C_2 + \cdots + C_r$ (resp. $\pi^{-1}(D) = D_1 + D_2 + \cdots + D_r$).

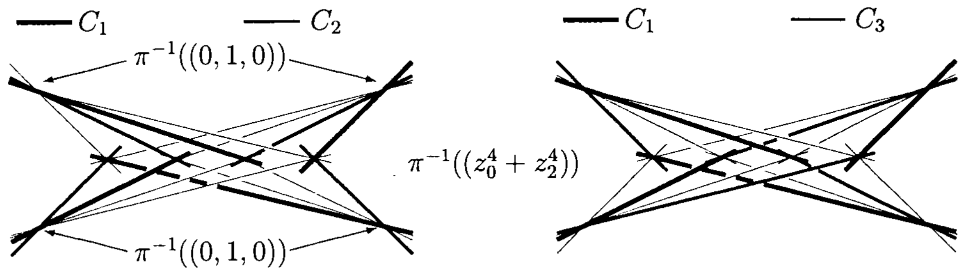
Lemma 6. *$C_i \cap D_j = \emptyset$, if $i \neq j$.*

Proof. $C_i \cap D_j \subset (\pi^*g_0 - \rho^i\tilde{f}\pi^*h) \cap (\pi^*g_0 - \rho^j\tilde{f}\pi^*h) \subset (\pi^*g_0)$. On the other hand, $C_i \subset (\pi^*g_2)$ and $D_j \subset (\pi^*g_1)$. Hence $C_i \cap D_j \subset (\pi^*g_0) \cap (\pi^*g_1) \cap (\pi^*g_2) = \emptyset$. □

Since $\left(\pi^*g_0 - \rho^i\tilde{f}\pi^*h\right) = C_i + D_i$ and $(\pi^*g_1) = D_1 + D_2 + \cdots + D_r$, we see that $\phi_I^{-1}(0) = C_{i_1} + C_{i_2} + \cdots + C_{i_p}$ and $\phi_I^{-1}(\infty) = D_{j_1} + D_{j_2} + \cdots + D_{j_q}$, where $\{j_1, j_2, \ldots, j_q\} = \{1, 2, \ldots, r\} \setminus I$ (see the figure in Introduction). Hence we see by the above lemma that the meromorphic map $\phi_I : X \to \mathbf{P}^1$ is holomorphic. In the following, we examine the structure of $\phi_I^{-1}(0)$ and $\phi_I^{-1}(\infty)$. First, we consider the case that $m = s$, i.e., h is a constant. Let x be a point in $C \setminus (f)$ and let $\pi^{-1}(x) = \{x_1, x_2, \ldots, x_r\}$. Then x_i is contained in only one among $C_1, C_2, \ldots, C_r$, because $C_j \subset (\pi^*g_0 - \rho^j\tilde{f}\pi^*h)$ and $(\pi^*g_0 - \rho^j\tilde{f}\pi^*h) \cap (\pi^*g_0 - \rho^k\tilde{f}\pi^*h) \subset (\pi^*f)$, if $j \neq k$. Conversely, C_j contains only one in $\{x_1, x_2, \ldots, x_r\}$, because if $x_i \in C_j$, then $\sigma^{j-k}x_i \in \{x_1, x_2, \ldots, x_r\} \cap C_k$. Hence the restriction $\pi_{|C_i} : C_i \to C$ of π to C_i is one to one. For example, if $f = z_0^r + z_1^r + z_2^r$, $g_0 = z_0$, $g_1 = z_1 - \epsilon z_2$ and $g_2 = (z_1 - \epsilon^3 z_2)\cdots(z_1 - \epsilon^{2r-1}z_2)$, where $\epsilon = \exp(2\pi\sqrt{-1}/2r)$, then $\phi_{\{1\}}^{-1}(0) = C_1$ and $\phi_{\{1\}}^{-1}(\infty) = D_2 + D_3 + \cdots + D_r$ consist of $r - 1$ rational curves meeting at a point. While if $n = r = 4$, $g_0 = z_0$, $g_1 = z_1^2 + \sqrt{-1}z_2^2$ and $g_2 = z_1^2 - \sqrt{-1}z_2^2$, then $\phi_{\{1,2\}}$ is an elliptic fibration and $\phi_{\{1,2\}}^{-1}(0) = C_1 + C_2$, $\phi_{\{1,2\}}^{-1}(\infty) = D_3 + D_4$ are singular fibers of type I_4.



Next, we consider the case that h is not a constant. As we see in the above, the restriction $\pi_{|C_i\setminus(\pi^*h)} : C_i \setminus (\pi^*h) \to C \setminus (h)$ of π to $C_i \setminus (\pi^*h)$ is one to one. Let x be a point in $(h)\cap(g_0)\cap C\setminus(f)$. Then C_i contains all points in $\pi^{-1}(x)$, because $C_i = (\pi^*g_0 - \rho^i\tilde{f}\pi^*h)\cap \pi^{-1}(C)$. Hence C_i and C_j cross each other at all points in $\pi^{-1}(x)$. For example, if $n = r = 4$, $-f = z_0^4 + z_1^4 + z_2^4$, $h = z_0$, $g_0 = z_1z_2$, $g_1 = -z_0^4 - z_1^4$ and $g_2 = z_0^4 + z_2^4$, then $(h)\cap(g_0)\cap C = \{(0,1,0)\}$. Hence we see that $\phi_{\{1,2\}}^{-1}(0) = C_1 + C_2$, $\phi_{\{1,2\}}^{-1}(\infty) = D_3 + D_4$ are singular fibers of type I_8 and that $\phi_{\{1,3\}}^{-1}(0) = C_1 + C_3$, $\phi_{\{1,3\}}^{-1}(\infty) = D_2 + D_4$ are disjoint union of two singular fibers of type I_4.



Proposition 7. *If $\phi_I^{-1}(0)$ or $\phi_I^{-1}(\infty)$ is connected, then all the fibers of ϕ_I are connected and the genus of generic fibers of ϕ_I is equal to $mpq(n - n/r - 3)/2 + 1$.*

Proof. Since $\phi_I^{-1}(0)$ and $\phi_I^{-1}(\infty)$ are reduced, $\phi_I^{-1}(t)$ are connected for all t in $\mathbf{P}^1$. First, we consider the case that $H := \{\tau \in \mathrm{Gal}(X/\mathbf{P}^2) | \phi_I\tau = \phi_I\} = 1$. Let $F = \phi_I^{-1}(t)$ be a generic fiber of ϕ_I. Then $((\pi^*g_0 - \rho^{i_1}\tilde{f}\pi^*h)(\pi^*g_0 - \rho^{i_2}\tilde{f}\pi^*h)\cdots(\pi^*g_0 - \rho^{i_p}\tilde{f}\pi^*h) - t\pi^*g_1) = F + D_{i_1} + D_{i_2} + \cdots + D_{i_p}$. Hence $\sum_{i=0}^{r-1}\sigma^i((\pi^*g_0 - \rho^{i_1}\tilde{f}\pi^*h)(\pi^*g_0 - \rho^{i_2}\tilde{f}\pi^*h)\cdots(\pi^*g_0 - \rho^{i_p}\tilde{f}\pi^*h) - t\pi^*g_1) = \pi^{-1}(\pi(F)) + p(D_1 + D_2 + \cdots + D_r)$. Therefore, the degree d of $\pi(F)$ is equal to $mpr - \deg(g_1)p = mpq$. On the other hand, $\sum_{1\leq i<j\leq r}(\sigma^iF)(\sigma^jF) = rd^2/2$, because $(\sigma_iF)^2 = 0$ and $\left(\sum_{i=0}^{r-1}\sigma^iF\right)^2 = rd^2$. Hence the number of the nodes of $\pi(F)$ is equal to $(rd^2/2 - sd\binom{r}{2})/r = d(d - sr + s)/2$, because $\pi(F)$ is tangent to (f) of order r at $nd/r = sd$ points. Therefore, the genus of $\pi(F)$ is equal to $(d-1)(d-2)/2 - d(d - sr + s)/2 = d(sr - s - 3)/2 + 1$. Also in the case that $H \neq 1$, we obtain the same result, using the Riemann-Hurwitz formula. □

In the case that $n = mr$ and $p = 1$, we can see what singular fibers appear on the fibration $\phi_{\{1\}}$ besides $\phi_{\{1\}}^{-1}(0)$ and $\phi_{\{1\}}^{-1}(\infty)$.

Proposition 8. *When $p = 1$, the images $\pi(F)$ of fibers $F = \phi_{\{1\}}^{-1}(t)$ of $\phi_{\{1\}}$ are defined by $((g_0 - tg_1)^r - (g_0^r + g_1g_2))/g_1 = 0$. In addition, if h is a constant and (f) is non-singular, then the restrictions $\pi_{|F} : F \to \pi(F)$ of π to F is biholomorphic for any t in $\mathbf{C} = \mathbf{P}^1\setminus\{\infty\}$.*

Proof. As we see in the proof of the above proposition, $\pi(F) + D$ is defined by $\prod_{i=0}^{r-1}(g_0 - tg_1 - \rho^i\tilde{f}h) \equiv (g_0 - tg_1)^r - (g_0^r + g_1g_2) = 0$. When h is a constant, the restrictions $\pi_{|F\setminus\pi^{-1}((f))} : F \setminus \pi^{-1}((f)) \to \pi(F)\setminus(f)$ of π to $F\setminus\pi^{-1}((f))$ are one to one and locally biholomorphic. Moreover, for each point x in $\pi(F)\cap(f)$ the inverse images

$\pi_{|F}^{-1}(x)$ under the restrictions $\pi_{|F}$ consist of one point and $\pi(F)$ is non-singular at x, if (f) is non-singular, because $f = (g_0 - tg_1)^r + g_1 g'$ and $(g_0 - tg_1)(x) = 0$, where g' is a defining equation of $\pi(F)$. □

Example 1. Let $n = r = 4$, let $p = 1$, let $g_0 = z_0$, let $g_1 = z_1 - \epsilon z_2$ and let $g_2 = (z_1^4 + z_2^4)/g_1$, where $\epsilon = \exp(2\pi\sqrt{-1}/8)$. Recall that $\phi_{\{1\}}$ is an elliptic fibration and that $\phi_{\{1\}}^{-1}(0)$ and $\phi_{\{1\}}^{-1}(\infty)$ are singular fibers of type IV. Let t be any point in $\mathbf{C}$ with $t^8 = 1$. Then we see by an easy calculation that $(((z_0 - t(z_1 - \epsilon z_2))^4 - (z_0^4 + z_1^4 + z_2^4))/(z_1 - \epsilon z_2))$ consists of the line $(z_0 - tz_1)$ (resp. $(z_0 + t\epsilon z_2)$) and a conic crossing at two points, if $t^4 = -1$ (resp. 1). Hence $\phi_{\{1\}}^{-1}(t)$ is a singular fiber of type I_2.

In the following, we show that any fibration on X the genus of whose generic fibers is equal to $n(r-1)(n - n/r - 3)/(2r) + 1$, coincides with one of holomorphic maps $\phi_{\{i\}}$ in Theorem 1.

Proposition 9. *Let $\phi : X \to B$ be a fibration on X and let F be a generic fiber of ϕ. Assume that $n - n/r - 3 > 0$. Then*

$$g(F) \geq n\frac{r-1}{2r}\left(n - \frac{n}{r} - 3\right) + 1.$$

The equality holds if and only if $H := \{\tau \in \mathrm{Gal}(X/\mathbf{P}^2) | \phi\tau = \phi\} = 1$ and $\deg(\pi(F)) = n(r-1)/r$. Moreover, then $\pi(F)$ is non-singular.

Proof. First, we note that τF is a fiber of ϕ, if and only if τ is in H for each element τ in $\mathrm{Gal}(X/\mathbf{P}^2)$, because each point in $\pi^{-1}((f))$ is a fixed point of τ and $\pi^{-1}((f))F > 0$. Let $X' = X/H$, let $q : X \to X'$ be the quotient map and let $\pi' : X' \to \mathbf{P}^2$ be the holomorphic map satisfying $\pi' \circ q = \pi$. Then there exists a fibration $\phi' : X' \to B$ on X' with $\phi' \circ q = \phi$. Let d be the degree of $\pi(F)$ and let $r' = r/|H|$ $(= |\mathrm{Gal}(X'/\mathbf{P}^2)|)$. Then $\pi(F)$ is tangent to (f) of order r' at dn/r' points, because $q(F)$ crosses $(\pi')^{-1}((f))$ normally. Let $(f) \cap \pi(F) = \{x_1, x_2, \dots, x_{dn/r'}\}$. Then x_j may be a cusp of $\pi(F)$ for each j. Let $\tilde{\sigma}(x_j) = 2\sum_{1 \leq i < j \leq r'} t_{ij}$, if $(\tau')^i q(F)$ are tangent to $(\tau')^j q(F)$ at $(\pi')^{-1}(x_j)$ of order t_{ij}, where τ' is a generator of $\mathrm{Gal}(X'/\mathbf{P}^2)$. Then the number of the nodes on $\pi(F)$ is equal to $(d^2 r' - \sum_{j=1}^{dn/r'} \tilde{\sigma}(x_j))/(2r') (\leq (d^2 - dn(r'-1)/r')/2)$. Hence $d \geq n(r'-1)/r'$ and the equality holds only if $\tilde{\sigma}(x_j) = r'(r'-1)$, i.e., $\pi(F)$ is non-singular at x_j for all j. Let (z_0, z_1) and (Z_0, Z_1) be local coordinate systems on neighborhoods U and V of x_j and $(\pi')^{-1}(x_j)$, respectively, satisfying $(\pi'_{|V})^* z_0 = Z_0^{r'}$, $(\pi'_{|V})^* z_1 = Z_1$, $z_0(x_j) = z_1(x_j) = 0$ and $(f) \cap U = (z_0)$. Then there exists a holomorphic function $h(Z_0)$ such that $q(F) \cap V$ is defined by $Z_1 - h(Z_0) = 0$. Since τ' is a generator of $\mathrm{Gal}(X'/\mathbf{P}^2)$, we see that $(\tau')^* Z_0 = \rho Z_0$, where ρ is a primitive r'-th root of the unit. Let $G(Z_0, Z_1) = \prod_{i=0}^{r'-1}(Z_1 - h(\rho^i Z_0))$. Then $(\pi')^{-1}(\pi(F)) (= \sum_{i=0}^{r'-1} (\tau')^i q(F))$ and $\pi(F)$ are locally defined by $G(Z_0, Z_1) = 0$ and $g(z_0, z_1) = 0$, respectively, where g is the holomorphic function satisfying $G(Z_0, Z_1) = g(Z_0^r, Z_1)$. Hence $\tilde{\sigma}(x_j) = \dim_{\mathbf{C}} \mathbf{C}[[Z_0, Z_1]]/(G, G_{Z_1}) = r' \dim_{\mathbf{C}} \mathbf{C}[[z_0, z_1]]/(g, g_{z_1})$. Let $\sigma(x_j) = \dim_{\mathbf{C}} \mathbf{C}[[z_0, z_1]]/(g, g_{z_1}) - r' + 1$. Since $g(0, z_1) = z_1^{r'}$, the genus of $q(F)$ is equal to

$$\frac{(d-1)(d-2)}{2} - \frac{d^2 - \sum_{j=1}^{dn/r'} \tilde{\sigma}(x_j)/r'}{2} - \sum_{j=1}^{dn/r'} \frac{\sigma(x_j)}{2} = \frac{d(n-n/r'-3)}{2} + 1$$

(see Theorem 2.1.9 in [2]). Since $\deg(q) = r/r'$ and $q_{|F}$ ramifies at dn/r' points in $(\pi')^{-1}((f) \cap \pi(F))$,

$$\begin{aligned} 2g(F) - 2 &= \frac{r}{r'}(2g(q(F)) - 2) + \frac{dn}{r'}\left(\frac{r}{r'} - 1\right) = \frac{r}{r'}d\left(n - \frac{n}{r'} - 3\right) + \frac{dr}{r'}\left(\frac{n}{r'} - \frac{n}{r}\right) \\ &= \frac{dr}{r'}\left(n - \frac{n}{r} - 3\right) \geq \frac{n}{(r')^2}(r'-1)r\left(n - \frac{n}{r} - 3\right). \end{aligned}$$

On the other hand, $(r'-1)/(r')^2 \geq (r-1)/r^2$ and the equality holds if and only if $r' = r$, because $2 \leq r' \leq r$. Now assume that the equality of the inequality in the proposition holds. Then $r = r'$ and $d = n(r-1)/r$. Hence $\pi(F)$ has no nodes and is non-singular at x_j for all j. Therefore, $\pi(F)$ is non-singular. □

Theorem 10. *Let $\phi : X \to B$ be a fibration on X. Assume that $n - n/r - 3 > 0$ and that the equality of the inequality in the above proposition holds. Then there exist homogenous polynomials g_0, g_1 and g_2 of degree n/r, n/r and $n(r-1)/r$, respectively, satisfying $f = g_0^r + g_1 g_2$, and $\phi = \theta \circ \phi_{\{i\}}$ for an integer i and for a biholomorphic map $\theta : \mathbf{P}^1 \to B$, where $\phi_{\{i\}}$ are the holomorphic maps in Theorem 1 for these polynomials g_0, g_1, g_2.*

Proof. Let F be a generic fiber of ϕ, let $D = \pi(F)$ and let g_2 be a defining equation of D. Then $\deg(g_2) = n(r-1)/r$ by the above proposition. Hence there exists the following short exact sequence of sheaves for each positive integer k.

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}\left(k\frac{n}{r} - \frac{n}{r}(r-1)\right) \longrightarrow \mathcal{O}_{\mathbf{P}^2}\left(k\frac{n}{r}\right) \longrightarrow \mathcal{O}_D\left(k\frac{n}{r}\right) \longrightarrow 0$$

Since $H^1(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}\left(k\frac{n}{r} - \frac{n}{r}(r-1)\right)) = 0$, we have the following exact sequence.

$$0 \longrightarrow \Gamma(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}\left(k\frac{n}{r} - \frac{n}{r}(r-1)\right)) \xrightarrow{g_2^*} \Gamma(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}\left(k\frac{n}{r}\right)) \xrightarrow{i^*} \Gamma(D, \mathcal{O}_D\left(k\frac{n}{r}\right)) \longrightarrow 0$$

Here the second arrow g_2^* sends g to $g_2 g$. Since D is non-singular, the restriction $\pi_{|F} : F \to D$ of π to F is biholomorphic. Hence there exists an element f' in $\Gamma(D, \mathcal{O}_D(n/r))$ with $\pi_{|F}^* f' = \tilde{f}_{|F}$. Then $(f')^r = i^*(f)$ and there exists an element g_0 in $\Gamma(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(n/r))$ with $i^*(g_0) = f'$, by the above exact sequence for $k = 1$. Since $i^*(f - g_0^r) = 0$, there exists an element g_1 in $\Gamma(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(n/r))$ with $g_2^*(g_1) = f - g_0^r$, again by the above exact sequence for $k = r$. On the other hand, the fiber $\phi_{\{i\}}^{-1}(0)$ of $\phi_{\{i\}}$ over 0 is an irreducible component of $\pi^{-1}(D)$ for each i. Hence $\phi_{\{i\}}^{-1}(0) = F$ for an integer i. Then $\phi = \theta \circ \phi_{\{i\}}$ for a biholomorphic map $\theta : \mathbf{P}^1 \to B$. □

Let $n = r$. If X has a fibration the genus of whose generic fibers is equal to $(n-1)(n-4)/2 + 1$, then there exists a line tangent to (f) of order n at a point, by the above theorem. Conversely, if there exists such a line (g_1), then we see in a way similar to the proof of the above theorem that there exist homogenous polynomials g_0 and g_2 satisfying $f = g_0^n + g_1 g_2$. Thus we see that the number of fibrations on X the genus of whose generic fibers is equal to $(n-1)(n-4)/2+1$, is equal to n times the number of lines tangent to (f) of order n at a point.

2 Proof of Theorem 2

Since there exists a surjective holomorphic map from B to $\mathbf{P}^1$, we may assume that $B = \mathbf{P}^1$. Let σ be the generator of $\mathrm{Gal}(Y/Z)$.

Lemma 11. $\nu := (\phi, \phi\sigma) : Y \to \mathbf{P}^1 \times \mathbf{P}^1$ *is a surjective holomorphic map.*

Proof. By the Stein factorization theorem, there exist a surjective holomorphic map $\tilde{\phi} : Y \to \tilde{B}$ whose fibers are connected and a finite holomorphic map $\theta : \tilde{B} \to \mathbf{P}^1$ with $\theta \circ \tilde{\phi} = \phi$. Suppose that $\tilde{\phi}(\sigma(\tilde{\phi}^{-1}(t_0)))$ is a point for a point t_0 in $\tilde{B}$. Then $\tilde{\phi}(\sigma(\tilde{\phi}^{-1}(t)))$ are also points for all points t in $\tilde{B}$, because $\sigma(\tilde{\phi}^{-1}(t_0))\sigma(\tilde{\phi}^{-1}(t)) = 0$. Hence the following commutative diagram holds.

$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & Y \\ \tilde{\phi}\downarrow & & \downarrow\tilde{\phi} \\ \tilde{B} & \xrightarrow{\exists\sigma'} & \tilde{B} \end{array}$$

Then there exists a surjective holomorphic map from $Z = Y/\langle\sigma\rangle$ to the curve $\tilde{B}/\langle\sigma'\rangle$. Hence $\tilde{\phi}(\sigma(\tilde{\phi}^{-1}(t))) = \tilde{B}$ for all points t in $\tilde{B}$. Therefore $\tilde{\nu} = (\tilde{\phi}, \tilde{\phi}\sigma) : Y \to \tilde{B} \times \tilde{B}$ is surjective. Since $(\theta, \theta) : \tilde{B} \times \tilde{B} \to \mathbf{P}^1 \times \mathbf{P}^1$ is surjective, $\nu = (\theta, \theta) \circ \tilde{\nu}$ is also surjective. □

The following commutative diagram holds.

$$\begin{array}{ccccc} Y & \xrightarrow{\nu} & \mathbf{P}^1 \times \mathbf{P}^1 & \ni & (\xi, \eta) \\ \downarrow\sigma & & \downarrow & & \downarrow \\ Y & \xrightarrow{\nu} & \mathbf{P}^1 \times \mathbf{P}^1 & \ni & (\eta, \xi) \end{array}$$

Hence the following also holds.

$$\begin{array}{cccc} Y & \xrightarrow{\nu} & \mathbf{P}^1 \times \mathbf{P}^1 & ((\xi_0, \xi_1), (\eta_0, \eta_1)) \\ \downarrow\lambda & & \downarrow & \downarrow \\ Z & \xrightarrow{\mu} & \mathbf{P}^2 & (\xi_0\eta_1 + \xi_1\eta_0, -2\xi_0\eta_0, 2\xi_1\eta_1) \end{array}$$

The map μ is surjective, by the above lemma. Since $\Delta_\lambda = \lambda(\{y \in Y | \sigma(y) = y\})$, $\mu(\Delta_\lambda) = [\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2]((\xi_0\eta_1 - \xi_1\eta_0)) = (w_0^2 + w_1 w_2)$.

3 Fibrations on double coverings of $\mathbf{P}^2$ ramifying along quartic curves

Let f be a reduced homogenous polynomial of z_0, z_1, z_2 of degree 4. Let $Y = X_{f,4}$ and let $\lambda : Y \to \mathbf{P}^2$ be the projection. Then Y is biholomorphic to the hypersurface of $\mathbf{P}^2$

defined by $z_3^4 - f(z_0, z_1, z_2) = 0$ and λ is a $\mathbf{Z}_4$-covering ramifying along each irreducible component of the curve $C := (f)$ with the ramification index 4. Let $\lambda_1 : X \to \mathbf{P}^2$ be a double covering ramifying along the curve C. Then there exists a holomorphic map $\lambda_2 : Y \to X$ satisfying $\lambda = \lambda_1 \circ \lambda_2$, which is a double covering ramifying along $\lambda_1^{-1}(C)$. If X has a fibration $\phi : X \to B$, then the base space B is a rational curve, because X is a rational surface. Hence we may assume that fibers of the composite $X \to \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ in Theorem 2 are connected. Then the pull-backs under μ of generic tangents of the conic $(w_0^2 + w_1 w_2)$ are irreducible.

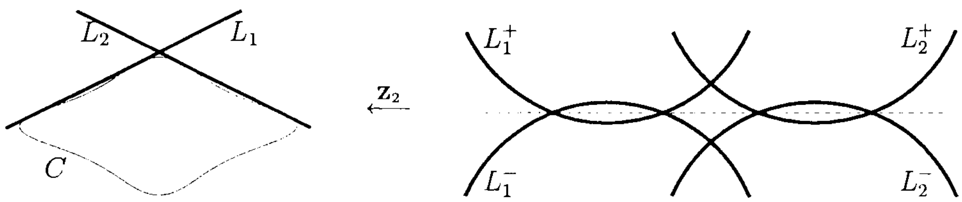
Proposition 12. *Assume that X has a fibration $\phi : X \to B$. Let $\mu = (g_0, g_1, g_2) : \mathbf{P}^2 \to \mathbf{P}^2$ be the holomorphic map in Theorem 2 satisfying $\pi \circ (\phi, \phi\sigma) = \mu \circ \lambda_1$, where $\langle \sigma \rangle = \mathrm{Gal}(X/\mathbf{P}^2)$. If the pull-backs under μ of generic tangents of the conic $(w_0^2 + w_1 w_2)$ are irreducible, then g_0, g_1, g_2 are quadratics.*

Proof. Since the restriction of the canonical divisor of Y to $Y \setminus \lambda^{-1}(\mathrm{Sing}(C))$ is trivial, generic fibers of $\phi \circ \lambda_2$ are elliptic curves. Let F be a generic fiber of ϕ. Then F crosses $\lambda_1^{-1}(C)$ normally. Hence the restriction $\lambda_{2|\lambda_2^{-1}(F)} : \lambda_2^{-1}(F) \to F$ of the double covering λ_2 to $\lambda_2^{-1}(F)$ ramifies at $F \cap \lambda_1^{-1}(C)$. Since $\lambda_2^{-1}(F)$ is an elliptic curve, $\lambda_{2|\lambda_2^{-1}(F)}$ ramifies at 4 points, because the ramification index of λ_2 along $\lambda_1^{-1}(C)$ is equal to 2. Namely, F crosses $\lambda_1^{-1}(C)$ at 4 points. Hence $\lambda_1(F)$ is tangent to C at 4 points. Therefore, $\lambda_1(F)$ is a quadric, because C is a quartic curve. On the other hand, $\lambda_1(F)$ is the pull-back of a tangent of the conic $(w_0^2 + w_1 w_2)$ under the holomorphic map defined by (g_0, g_1, g_2). Hence g_0, g_1, g_2 are quadratics. □

When C is a generic quartic curve, we see by the following proposition that X has a fibration, i.e., there exist quadratics g_0, g_1, g_2 satisfying $f = g_0^2 + g_1 g_2$ and $(g_0) \cap (g_1) \cap (g_2) = \emptyset$.

Proposition 13. *Let L_1 and L_2 be bitangents of the quartic curve C. Then there exist exactly two fibrations $\phi : X \to \mathbf{P}^1$ with a fiber F satisfying $\lambda_1(F) = L_1 + L_2$.*

Proof. Since L_i are simply connected and the restrictions of λ_1 to $\lambda_1^{-1}(L_i)$ are unramified, $\lambda_1^{-1}(L_i)$ consist of two irreducible components. Let $\lambda_1^{-1}(L_i) = L_i^+ + L_i^-$. Since $L_i^+ L_i^- = 2$ and $(L_i^+ + L_i^-)^2 = 2$, we see that $(L_i^{\pm})^2 = -1$. We may assume that $L_1^+ L_2^+ > 0$, because $(L_1^+ + L_1^-)(L_2^+ + L_2^-) = 2$. Then $L_1^+ L_2^+ = L_1^- L_2^- = 1$ and $L_1^+ L_2^- = L_1^- L_2^+ = 0$. Hence X has two fibrations with $L_1^{\pm} + L_2^{\pm}$ as a fiber. On the other hand, we easily see that there does not exist a fibration with $F = L_1^{\pm} + L_2^{\mp}$, $L_1^+ + L_1^- + L_2^{\pm}$ or $L_1^{\pm} + L_2^+ + L_2^-$ as a fiber. □



Let L_1, L_2, L_3 and L_4 be bitangents of C. Then we see by Theorem 2 and Proposition 12 that there exists a fibration ϕ on X with fibers F_1 and F_2 satisfying $\lambda_1(F_1) = L_1 + L_2$ and $\lambda_1(F_2) = L_3 + L_4$, if and only if there exists a holomorphic map $\mu : \mathbf{P}^2 \to \mathbf{P}^2$ such that $\mu(L_1 + L_2)$ and $\mu(L_3 + L_4)$ are tangents of the conic $(w_0^2 + w_1w_2)$ and that $\mu^*(w_0^2 + w_1w_2) = (f)$, i.e., there exist quadratics g_0, g_1 and g_2 satisfying $(g_1) = L_1 + L_2$, $(g_2) = L_3 + L_4$, $f = g_0^2 + g_1g_2$ and $(g_0) \cap (g_1) \cap (g_2) = \emptyset$. For example, $(z_0^4 + z_1^4 + z_2^4)$ has four bitangents $(z_0 \pm z_1 \pm z_2)$. Since $(z_0^2 + z_1^2 + z_2^2)^2 + (z_0 + z_1 + z_2)(z_0 + z_1 - z_2)(z_0 - z_1 + z_2)(z_0 - z_1 - z_2) = 2(z_0^4 + z_1^4 + z_2^4)$, if $f = z_0^4 + z_1^4 + z_2^4$, then X has a fibration with fibers F_1 and F_2 satisfying $\lambda_1(F_1) = (z_0 + z_1 + z_2) + (z_0 + z_1 - z_2)$ and $\lambda_1(F_2) = (z_0 - z_1 + z_2) + (z_0 - z_1 - z_2)$.

Proposition 14. *Let L_1, L_2, L_3 and L_4 be bitangents of C. Assume that there exists a fibration $\phi : X \to \mathbf{P}^1$ on X with fibers F_1 and F_2 satisfying $\lambda_1(F_1) = L_1 + L_2$ and $\lambda_1(F_2) = L_3 + L_4$. Then there also exists a fibration $\psi : X \to \mathbf{P}^1$ with fibers G_1 and G_2 satisfying $\lambda_1(G_1) = L_1 + L_3$ and $\lambda_1(G_2) = L_2 + L_4$. Moreover, if there exist other bitangents L_5, L_6 of C and a fiber F_3 of ϕ satisfying $\lambda_1(F_3) = L_5 + L_6$, then any irreducible component of $\lambda_1^{-1}(L_5)$ does not contained in a fiber of ψ.*

Proof. Let $\lambda_1^{-1}(L_i) = L_i^+ + L_i^-$ $(1 \leq i \leq 6)$ and assume that $L_1^+ + L_2^+$, $L_3^+ + L_4^+$ and $L_5^+ + L_6^+$ are fibers of ϕ. Then $L_1^+L_3^- = L_2^-L_4^+ = 1$ and $L_1^+L_2^- = L_1^+L_4^+ = L_3^-L_2^- = L_3^-L_4^+ = 0$. Hence $(L_1^+ + L_3^-)^2 = (L_2^- + L_4^+)^2 = (L_1^+ + L_3^-)(L_2^- + L_4^+) = 0$. Therefore, there exists a fibration $\psi : X \to \mathbf{P}^1$ with $L_1^+ + L_3^-$ and $L_2^- + L_4^+$ as fibers. Since $(L_1^+ + L_3^-)L_5^+ = (L_1^+ + L_3^-)L_5^- = 1$, none of the irreducible components L_5^+ and L_5^- of $\lambda_1^{-1}(L_5)$ are contained in a fiber of ψ. □

In the following, we consider how many fibrations X has, restricting ourselves to the case that C has only nodes as singularities. Let $\{p_1, p_2, \ldots, p_l\}$ be the set of nodes of C. Let $\varpi : \widetilde{X} \to X$ be the minimal resolution of X and let $(\lambda_1 \circ \varpi)^{-1}(p_i) = E_i$. Then we easily see that $\chi(\widetilde{X}) = 10$ and that E_i are non-singular rational curves with the self-intersection number -2.

Remark There may be fibrations on $\widetilde{X}$ which are not pull-backs of those on X, i.e., at least one of E_1, E_2, ... and E_l is not contained in a fiber of which.

Lemma 15. *Assume that C has only nodes as singularities. Let $\phi : X \to \mathbf{P}^1$ be a fibration on X. Then the image of any fiber of ϕ under λ_1 is one of the following.*

(i) a smooth conic tangent to C of even order at smooth points of C

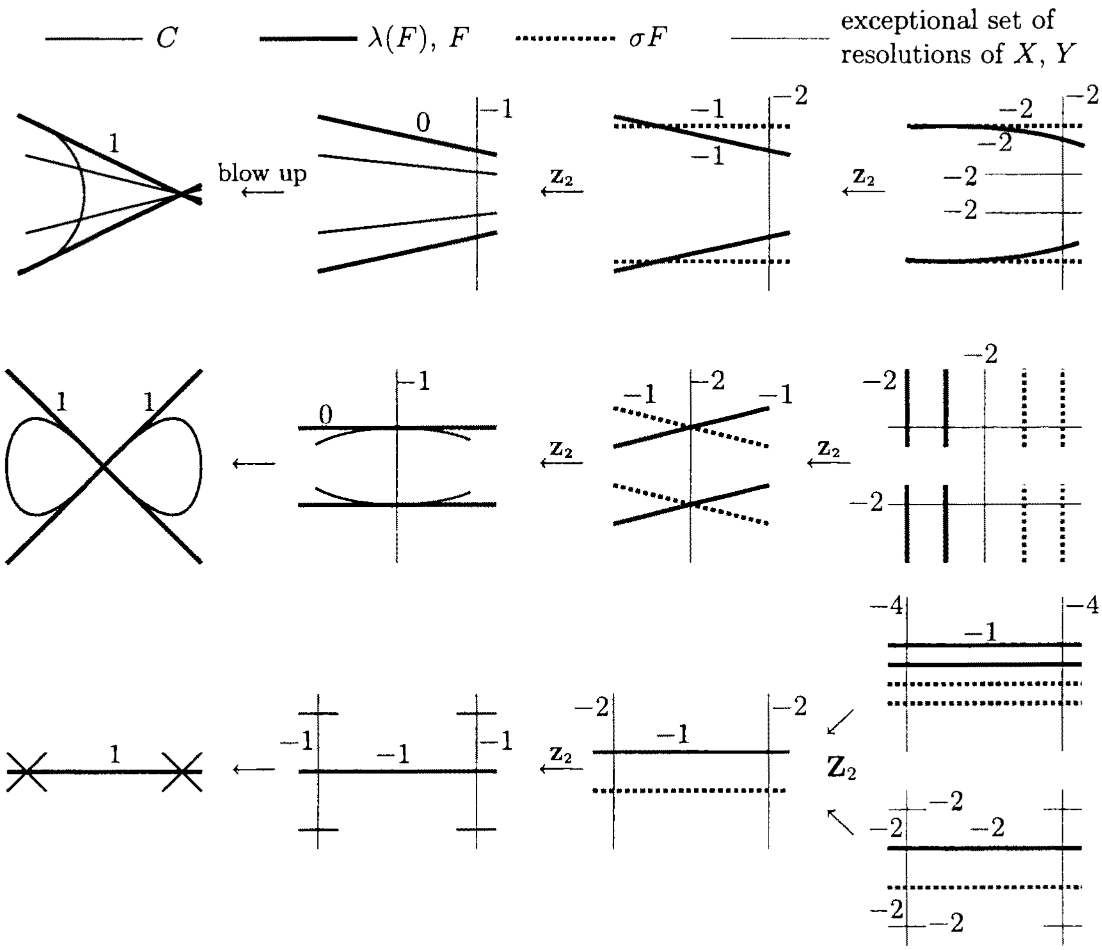
(ii) two lines tangent to C of even order at smooth points of C

(iii) two lines passing through one and the same node of C and tangent to C of order 2 at a smooth point of C (or tangent to one branch of C of order 3 at the node)

(iv) a line passing through two nodes of C

Proof. Let F be a fiber of ϕ and let F_1, F_2, ..., F_s be the irreducible components of F. Suppose that $\sigma F_i = F_i$ for an irreducible component F_i of F, where σ is the generator of $\mathrm{Gal}(X/\mathbf{P}^2)$. Then F_i is contained in a fiber of each of two fibrations ϕ and $\phi\sigma$.

Hence the image of F_i under the map $\nu = (\phi, \phi\sigma)$ is a point. Therefore, that under the composite $\pi \circ \nu = \mu \circ \lambda_1$ is also a point, where $\pi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2$ and $\mu : \mathbf{P}^2 \to \mathbf{P}^2$ are the maps in Theorem 2. However, μ and λ_1 are finite maps, a contradiction. Hence $\sigma F_i \neq F_i$. Let $M = \lambda_1(F)$. Then M is a quadric by Proposition 12. Hence M consists of a smooth conic, two lines or a double line. Therefore, the number s of the irreducible components of F is equal to 1 or 2. On the other hand, if M passes through nodes $p_{i_1}, p_{i_2}, \ldots$ and p_{i_k} of C, then $\varpi^{-1}(F) = c_1\widetilde{F_1} + \cdots + c_s\widetilde{F_s} + d_1E_{i_1} + \cdots + d_kE_{i_k}$, which is a fiber of the fibration $\phi \circ \varpi$ on $\widetilde{X}$, where $\widetilde{F_j}$ are the proper transformations of F_j under ϖ and c_i, d_j are positive numbers such that $F = c_1F_1 + \cdots + c_sF_s$.



First, we consider the case that M consists of a smooth conic. Since $\lambda_1^{-1}(M)$ consists of two curves, M is tangent to C of even order. We see by an easy calculation that the self-intersection number of the proper transformation of F under ϖ is equal to 0, even if M passes through nodes of C. Hence if M passes through at least one node of C, then F can not be a fiber of ϕ.

Next, let L be a line tangent to C of odd order at a point of C. Then $\lambda_1^{-1}(L)$ is irreducible and $\sigma\lambda_1^{-1}(L) = \lambda_1^{-1}(L)$. Hence L can not be an irreducible component of M.

Let L be a line tangent to C of even order. Then $\lambda_1^{-1}(L)$ consists of two curves, the proper transformation L_0 of each of which under ϖ has the self-intersection number -1, even if L passes through nodes of C (see the figures above). Moreover, if L passes through a node p_i, then L_0 intersect E_i at a point. Hence if M consists of a line, then M passes through exactly two nodes of C. While, if M consists of two lines, then both are bitangents of C which do not pass through nodes of C or both pass through one and the same node of C. □

Let M be a smooth conic tangent to C of even order and passing through nodes p_{i_1}, p_{i_2}, ..., p_{i_k}. Then as we see in the proof of the above lemma, the proper transformation under ϖ of each irreducible component of $\lambda_1^{-1}(M)$, is a rational curve with the self-intersection number 0. Hence there exists a fibration $\psi : \widetilde{X} \to \mathbf{P}^1$ with a fiber F satisfying $(\lambda_1 \circ \varpi)(F) = M$ and E_{i_1}, E_{i_2}, ..., E_{i_k} are sections of ψ.

Proposition 16. *If C is a non-singular irreducible quartic curve, then X has exactly* 126 *fibrations.*

Proof. Let $\phi : X \to \mathbf{P}^1$ be a fibration on X and let F be a fiber of ϕ. Then $M := \lambda_1(F)$ consists of a smooth conic tangent to C of even order or two bitangents of C, by the above lemma. In the former case, F is a non-singular rational curve with the self-intersection number 0. In the latter case, F consists of two rational curves with the self-intersection number -1 crossing each other at a point. On the other hand, $\chi(X) = 10$. Hence ϕ has exactly six fibers whose images under λ_1 consist of two bitangents of C. Since there exist 28 bitangents of C, we see by Proposition 13 that X has $2\binom{28}{2}/6 = 126$ fibrations. □

Proposition 17. *If C is an irreducible quartic curve with a node as singularities, then X has exactly* 60 *fibrations.*

Proof. Let ϕ be a fibration on X. Then by Lemma 15, if the image $\lambda_1(F)$ under λ_1 of a fiber F of ϕ passes through the node p_1 of C, then it consists of two lines passing through p_1 and tangent to C of even order at another point (or tangent to one branch of C of order 3 at the node p_1). Let L_1 and L_2 be such lines. Then the pull-backs of L_i under λ_1 consist of two irreducible components, we denote by $L_i^{\pm}$ the proper transformation of which under ϖ. Then we easily see that $E_1 L_i^{\pm} = L_i^{+} L_i^{-} = 1$, $(L_i^{\pm})^2 = -1$ and $L_1^{\pm} L_2^{\pm} = L_1^{\pm} L_2^{\mp} = 0$. Hence there exist four fibrations on $\widetilde{X}$ each of which has one of $E_1 + L_1^{+} + L_2^{+}$, $E_1 + L_1^{+} + L_2^{-}$, $E_1 + L_1^{-} + L_2^{+}$ and $E_1 + L_1^{-} + L_2^{-}$ as a fiber. Therefore, there exist exactly four fibrations ϕ on X with a fiber F satisfying $\phi(F) = L_1 + L_2$. On the other hand, we see by the Plüker formulas that there exist six lines passing through the node p_1 of C and tangent to C at another point (or tangent to one branch of C of order 3 at p_1). Hence we see that X has $4\binom{6}{2} = 60$ fibrations. □

Proposition 18. *If C is an irreducible quartic curve with two nodes as singularities, then X has exactly* 26 *fibrations.*

Proof. Let L be the line passing through the two nodes of C. We easily see that there exist two fibrations ϕ on X with a fiber F satisfying $\lambda_1(F) = L$. We also see as in the proof of the above proposition that the number of the other fibrations is equal to $4\binom{4}{2} = 24$. □

Example 2. Let C be the quartic curve on $\mathbf{P}^2$ defined by $z_0^4 + z_0^2z_1^2 + z_0^2z_2^2 - z_1^2z_2^2 = 0$. Then C has two nodes at $(0,1,0)$ and $(0,0,1)$. Four of 13 pairs of fibrations on a double covering of $\mathbf{P}^2$ ramifying along C are obtained by the holomorphic maps defined by

$$(2z_0^2 + z_1^2 + z_2^2, -z_1^2 + 2\sqrt{-1}z_1z_2 - z_2^2, z_1^2 + 2\sqrt{-1}z_1z_2 + z_2^2), \quad (\sqrt{2}z_0^2, -z_0^2 + z_1^2, z_0^2 - z_2^2),$$
$$((1 \pm \sqrt{2})z_0^2 + z_1^2, \quad z_0z_1 - z_1^2 - (\sqrt{2} \pm 1)\sqrt{-1}z_0z_2 + (\sqrt{2} \pm 1)\sqrt{-1}z_1z_2,$$
$$z_0z_1 + z_1^2 + (\sqrt{2} \pm 1)\sqrt{-1}z_0z_2 + (\sqrt{2} \pm 1)\sqrt{-1}z_1z_2).$$

Proposition 19. *If C is an irreducible quartic curve with three nodes as singularities, then X has exactly* 8 *fibrations.*

Proof. Let L_3 (resp. L_2) be the line passing through p_1 and p_2 (resp. p_1 and p_3). Then we easily see that there exist two fibrations ϕ on X with a fiber F satisfying $\lambda_1(F) = L_3$ (resp. L_2). The number of the other fibrations on X is equal to 4, because there exist two lines passing through the node p_1 and tangent to C of even order. □

Example 3. Let C be the quartic curve on $\mathbf{P}^2$ defined by $z_0^2z_1^2 + z_1^2z_2^2 + z_2^2z_0^2 = 0$. Then C has three nodes at $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and four bitangent $(z_0 \pm z_1 \pm z_2)$. Four pairs of the fibrations on a double covering of $\mathbf{P}^2$ ramifying along C are obtained by the holomorphic maps $\mu_i : \mathbf{P}^2 \to \mathbf{P}^2$, where $\mu_1 = (z_0^2, -z_0^2 - z_1^2, z_0^2 + z_2^2)$, $\mu_2 = (z_1z_2, z_0^2, z_1^2 + z_2^2)$, $\mu_3 = (z_2z_0, z_1^2, z_2^2 + z_0^2)$ and $\mu_4 = (z_0z_1, z_2^2, z_0^2 + z_1^2)$. The pull-backs $(z_0^2 + z_1^2)$, $(z_0^2 + z_2^2)$ and $(z_1^2 + z_2^2)$ under μ_1 of tangents (w_1), (w_2) and $(2w_0 + w_1 - w_2)$ of the conic $Q := (w_0^2 + w_1w_2)$ consist of two lines both of which cross C only at the nodes $(0,0,1)$, $(0,1,0)$ and $(1,0,0)$, respectively. Those of other tangents of Q consist of a smooth conic. While, the pull-backs under μ_2 of tangents $(2w_0 - w_1 + w_2)$ and $(2w_0 + w_1 - w_2)$ of the conic Q consist of the two bitangents $(z_0 \pm z_1 \pm z_2)$ and $(z_0 \pm z_1 \mp z_2)$, respectively, that of (w_1) consists of the line passing through the two node $(0,1,0)$ and $(0,0,1)$ of C, and that of (w_2) consists of two lines crossing C at the node $(1,0,0)$.

We can prove the following proposition in a way similar to that in the proof of the above propositions, noting the fact that for a point p on a non-singular cubic curve Q there exist four lines passing through p and tangent to Q at another point or tangent to Q of order 3 at p.

Proposition 20. *If C consists of a non-singular cubic curve (resp. a cubic curve with a node) and a line crossing each other at* 3 *points, then X has exactly* 24 *(resp.* 6*) fibrations.*

Example 4. Let $\mu : \mathbf{P}^2 \to \mathbf{P}^2$ be the holomorphic map defined by

$$(z_0^2 - z_0z_1 + z_1^2 + \sqrt[3]{2}z_2^2, \quad -z_0^2 + 2\rho z_0z_1 - \rho^2z_1^2 - \sqrt[3]{4}\rho^2z_0z_2 - \sqrt[3]{4}z_1z_2,$$
$$z_0^2 - 2\rho^5z_0z_1 + \rho^4z_1^2 + \sqrt[3]{4}\rho^4z_0z_2 + \sqrt[3]{4}z_1z_2),$$

where $\rho = \exp(2\pi\sqrt{-1}/6)$. Then the pull-back under μ of the conic $(w_0^2 + w_1w_2)$ consists of the non-singular cubic curve $(z_0^3 + z_1^3 + z_2^3)$ and the line (z_2). That of the tangent (w_1) (resp. (w_2), $(2w_0 + \rho w_1 - \rho^5 w_2)$) of $(w_0^2 + w_1w_2)$ consists of two lines $(z_0 - \rho z_1)$ and $(z_0 - \rho z_1 + \sqrt[3]{4}\rho^2 z_2)$ (resp. $(z_0 - \rho^5 z_1)$ and $(z_0 - \rho^5 z_1 + \sqrt[3]{4}\rho^4 z_2)$, $(z_0 + z_1 + \sqrt[3]{4}\rho^2 z_2)$ and $(z_0 + z_1 + \sqrt[3]{4}\rho^4 z_2)$). These two lines are tangent to $(z_0^3 + z_1^3 + z_2^3)$ and pass through the point $(\rho, 1, 0)$ (resp. $(\rho^5, 1, 0)$, $(-1, 1, 0)$), at which $(z_0^3 + z_1^3 + z_2^3)$ and (z_2) intersect.

In the case that f is the product of two quadratics q_1 and q_2, we can write explicitly the holomorphic map μ in Theorem 2. If there exist two linear equations l_1 and l_2 satisfying

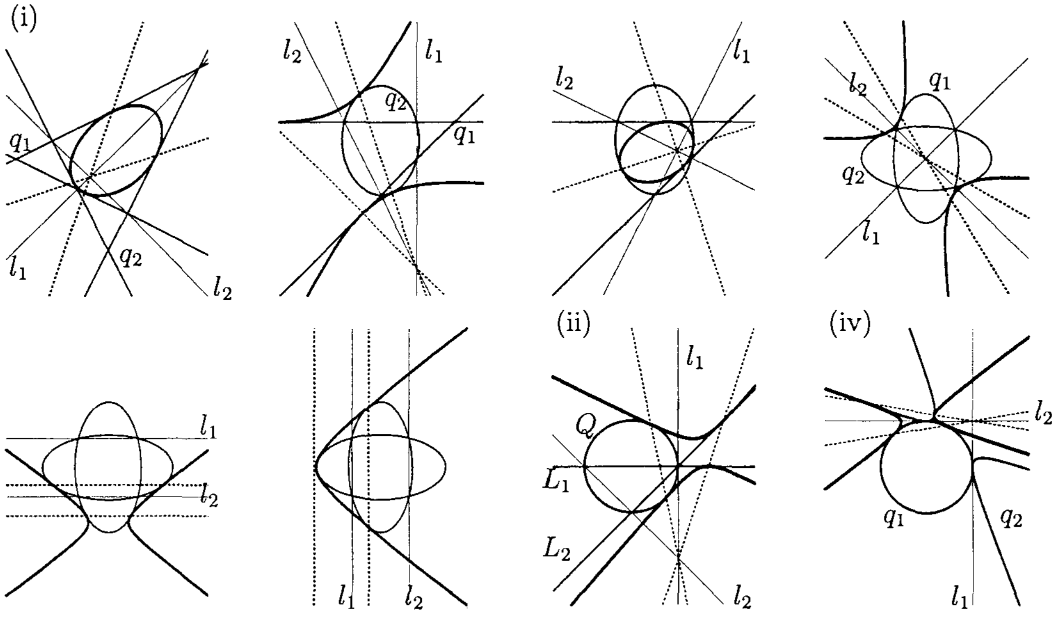
$$(*) \qquad q_1 - q_2 = l_1 l_2 \quad \text{and} \quad (q_1) \cap (q_2) \cap (l_1) \cap (l_2) = \emptyset,$$

then $\mu^*(w_0^2 + w_1w_2) = (q_1q_2)$, where $\mu : \mathbf{P}^2 \to \mathbf{P}^2$ is the holomorphic map defined by $(q_1 + q_2, -l_1^2, l_2^2)$. In the following cases, if we replace q_1 and q_2 by their suitable constant multiplications, then there exist linear equations l_1 and l_2 satisfying $(*)$.

(i) In the case that two quadrics (q_1) and (q_2) intersect each other at four points p_1, p_2, p_3 and p_4, let l_1 (resp. l_2) be a defining equation of the line passing through the two points p_1 and p_2 (resp. p_3 and p_4). Here we assume that if (q_1) or (q_2) consists of two lines, p_1 and p_2 (resp. p_3 and p_4) are not on one and the same line. Hence there exist one, two or three pairs $\{l_1, l_2\}$ satisfying $(*)$, accordingly as (q_1q_2) consists of four lines, a conic and two lines or two conics.

(ii) In the case that (q_1q_2) consists of a conic Q and two lines L_1, L_2 intersecting at a point p, let l_1 be a defining equation of the tangent of Q at p, and let l_2 be that of the line passing through the intersection points of Q with L_1 and L_2.

(iii) In the case that (q_1q_2) consists of two conics tangent at a point p_1 and crossing each other at two points p_2, p_3, let l_1 be a defining equation of the tangent of the two conics at p, and let l_2 be that of the line passing through the intersection points p_2, p_3.



(iv) In the case that (q_1q_2) consists of two conics tangent at two points p_1 and p_2, let l_1 and l_2 be defining equations of the tangents of the two conics at p_1 and p_2, respectively.

On the other hand, if $\mu : \mathbf{P}^2 \to \mathbf{P}^2$ is the holomorphic map defined by (g_0, g_1, g_2) for quadratics g_0, g_1 and g_2 satisfying $f = g_0^2 + g_1g_2$ and $(g_0) \cap (g_1) \cap (g_2) = \emptyset$, then the pull-back Q under μ of a generic tangent of the conic $(w_0^2 + w_1w_2)$ is a conic tangent to $C = (f)$ at four points. In particular, if f is the product of two quadratics q_1 and q_2, then Q is tangent to each of (q_1) and (q_2) at two points. Conversely, we have:

Proposition 21. *Let Q be a conic tangent to each of two quadrics (q_1) and (q_2) at two points. Then there exist linear equations l_1 and l_2 satisfying $(*)$ and Q is the pull-back of the tangent $(2w_0 + w_1 - w_2)$ of the conic $(w_0^2 + w_1w_2)$ under the holomorphic map $\mu : \mathbf{P}^2 \to \mathbf{P}^2$ defined by $(q_1 + q_2, -l_1^2, l_2^2)$, if we replace q_1 and q_2 by their certain constant multiplications.*

Proof. Let l'_1 (resp. l'_2) be a defining equation of the line passing through the two tangential points of Q and (q_1) (resp. (q_2)). Then we may assume that Q is defined by $q_1 - (l'_1)^2 = 0$ (resp. $q_2 - (l'_2)^2 = 0$), replacing l'_1 (resp. l'_2) by its constant multiplication. Moreover, we may assume that $q_1 - (l'_1)^2 \equiv q_2 - (l'_2)^2$, replacing q_2 and l'_2 by their constant multiplications. Then $l_1 := l'_1 + l'_2$ and $l_2 := l'_1 - l'_2$ satisfy $(*)$ and Q is defined by $2(q_1 + q_2) - l_1^2 - l_2^2 = 0$. □

Corollary 22. *The number of the fibrations on X is equal to* 6, 2, 4, 2 *or* 2, *accordingly as C consists of two conics crossing each other at four points, two conics tangent to each other at two points, a conic and two lines in a general position, a conic and two lines intersecting at a point or four lines in a general position.*

Assume that (q_1q_2) consists of four lines with at least one triple point, a conic and two lines with at least one tangent point or two conics tangent to each other of order 3 or 4 at a point. Then there do not exist l_1 and l_2 satisfying $(*)$. Hence there do not exist quadratics g_0, g_1 and g_2 satisfying $f = g_0^2 + g_1g_2$, by the above proposition.

4 Fibrations on $\mathbf{Z}_4$-coverings of $\mathbf{P}^2$ ramifying along quartic curves

Let $\lambda : Y \to \mathbf{P}^2$ be a $\mathbf{Z}_4$-covering ramifying along a quartic curve C and let σ be a generator of $\mathrm{Gal}(Y/\mathbf{P}^2)$. Here we note that if C is irreducible or consists of an irreducible cubic curve and a line, then $\mathbf{Z}_4$-coverings of $\mathbf{P}^2$ ramifying along C are unique up to isomorphisms, because $H_1(\mathbf{P}^2 \setminus C, \mathbf{Z}) = \mathbf{Z}_4$ or $\mathbf{Z}$ (see Proposition 1.3 in Chapter 4 of [1]). Then Y is biholomorphic to the hypersurface in $\mathbf{P}^3$ defined by $z_3^4 - f(z_0, z_1, z_2) = 0$, where f is a defining equation of C. While if C consists of four lines, two lines and a conic or two conics, then there exist at least two isomorphism classes of $\mathbf{Z}_4$-coverings of $\mathbf{P}^2$ ramifying along C. Throughout the rest of this section, we assume that the ramification index of λ along each irreducible component of C is equal to 4. Assume that Y has a fibration $\phi : Y \to \mathbf{P}^1$ and let F be a fiber of ϕ. Then we see as in the proof of Lemma 11 that

σF is not a fiber of ϕ. Moreover, $\sigma^2 F$ is a fiber of ϕ, if and only if $\phi \circ \sigma^2 = \phi$, because each point on $\lambda^{-1}(C)$ is a fixed point of σ^2. In the case that $\phi \circ \sigma^2 = \phi$ (resp. $\neq \phi$), the left (resp. right) of the following commutative diagrams holds, by Theorem 2.

$$\begin{array}{ccc} X & \xrightarrow{\nu_1} & \mathbf{P}^1 \times \mathbf{P}^1 \\ \downarrow \lambda_1 & & \downarrow \pi \\ \mathbf{P}^2 & \xrightarrow{\mu_1} & \mathbf{P}^2 \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\nu_2=(\phi,\phi\circ\sigma^2)} & \mathbf{P}^1 \times \mathbf{P}^1 \\ \downarrow \lambda_2 & & \downarrow \pi \\ X & \xrightarrow{\mu_2} & \mathbf{P}^2 \end{array}$$

Here $X = Y/\langle\sigma^2\rangle$ and $\lambda_2 : Y \to X$, $\lambda_1 : X \to \mathbf{P}^2$ are the double coverings induced from λ. By Theorem 2, the conic $(w_0^2 + w_1 w_2)$ is the image under μ_1 (resp. μ_2) of the quartic curve C (resp. $\lambda_1^{-1}(C)$). When C has only nodes as singularities, we know the number of the fibrations ϕ satisfying $\phi \circ \sigma^2 = \phi$, by Propositions 16, 17, 18, 19, 20 and Corollary 22 Assume that there exist homogenous polynomials g_0, g_1, g_2 and h satisfying $fh^4 = g_0^4 + g_1 g_2$, $(g_0) \cap (g_1) \cap (g_2) = \emptyset$ and $p := \deg(g_1)/\deg(g_0) \in \mathbf{Z}$. Then by Theorem 1 we have fibrations ϕ_I for all $I \subset \{1,2,3,4\}$ with $|I| = p$. If $p = 2$ and if $I = \{1,3\}$ or $\{2,4\}$, then $\phi \circ \sigma^2 = \phi$. In the other cases, $\phi \circ \sigma^2 \neq \phi$. When $p = 1$ and $I = \{4\}$ (resp. $p = 2$ and $I = \{4,1\}$), ϕ_I is defined by $(g_0 - \tilde{f}h, g_1)$ (resp. $((g_0 - \tilde{f}h)(g_0 - \sqrt{-1}\tilde{f}h), g_1)$. Hence μ_2 is defined by $(g_0 g_1, \tilde{f}^2 h^2 - g_0^2, g_1^2)$ (resp. $(g_0^2 + \sqrt{-1}\tilde{f}^2 h^2, g_2, g_1)$).

Let $\phi : Y \to \mathbf{P}^1$ be a fibration on Y satisfying $\phi \circ \sigma^2 = \phi$. In the following, we consider what singular fibers appear on the fibration $\phi \circ \varpi$ on $\tilde{Y}$, restricting ourselves to the case that C has only nodes as singularities, where $\varpi : \tilde{Y} \to Y$ is the minimal resolution of Y. Here we note that if Y is biholomorphic to the hypersurface in $\mathbf{P}^3$ defined by $z_3^4 - f(z_0, z_1, z_2) = 0$, then Y has only rational double points of type A_3 as singularities and $\tilde{Y}$ is a $K3$ surface. In the other cases, the exceptional set of ϖ contains rational curves with the self-intersection number -4 and $\tilde{Y}$ is a rational surface. Let F be a fiber of ϕ. Then as we see in Lemma 15 that $\lambda(F)$ consists of a smooth conic tangent to C of even order at smooth points, two lines tangent to C of even order at smooth points, two lines passing through one and the same node of C or a line passing through two nodes of C. We easily see that the relations in the following table hold (see the figures following Lemma 15).

$\lambda(F)$	How $\lambda(F)$ crosses C	type of $\varpi^{-1}(F)$
a smooth conic	$(2,2,2,2)$	I_0
	$(4,2,2)$	I_1
	$(4,4)$	I_2
	$(6,2)$	II
	(8)	III
two bitangents	$(2,2)$, $(2,2)$	I_2
	$(2,2)$, (4)	I_3
	(4), (4)	I_4
two lines passing through a node of C	$(2,2)$, $(2,2)$	I_0^*
	$(2,2)$, (4)	I_1^*
	(4), (4)	I_2^*
a line passing through two nodes of C	$(2,2)$	I_2^* or I_2

In the second column of the above table $(l, m, \ldots, n)$ implies that an irreducible component L of $\lambda(F)$ is tangent to C of order l at a point, of order m at another point, ... and of order n at another point. In particular, if L is a line passing through a node p of C, then $(2,2)$ (resp. (4)) implies that L is tangent to C of order 2 at another point (resp. tangent to one branch of C of order 3 at the node p). The third row of the above table occurs only in the cases that C consists of an irreducible quartic curve, an irreducible cubic curve and a line or a conic and two lines. In the first two cases Y has only rational double points as singularities. In the last case the node of C which $\lambda(F)$ pass through, is the intersection point of the two lines which are irreducible components of C. Then the inverse image of the node under λ, is a rational double point. On the other hand, the fourth row occurs also in the case that C consists of four lines or two conics. When Y is biholomorphic to the hypersurface in $\mathbf{P}^3$ defined by $z_3^4 - f(z_0, z_1, z_2) = 0$, the fiber F of ϕ whose image $\lambda(F)$ under λ is a line passing through two nodes of C, is of type I_2^*. In the other cases, the inverse images of the two nodes of C which $\lambda(F)$ pass through, may not be rational double points. Then $\varpi^{-1}(F)$ is of type I_2. If the map μ_1 in the commutative diagram at the beginning of this section, is defined by relatively simple polynomials, then we see as in the following examples which in the above table occur.

Example 5. Let $\lambda : Y \to \mathbf{P}^2$ be a $\mathbf{Z}_4$-covering ramifying along the quartic curve $(z_0^4 + z_1^4 + z_2^4)$. Let $\phi : Y \to \mathbf{P}^1$ be the fibration obtained as the composite of the above three horizontal arrows in the following commutative diagram.

$$\begin{array}{ccccccc}
Y & \longrightarrow & Y/\langle\sigma^2\rangle & \longrightarrow & \mathbf{P}^1 \times \mathbf{P}^1 & \xrightarrow{p_1} & \mathbf{P}^1 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbf{P}^2 & \xrightarrow{\mu} & \mathbf{P}^2 & & \\
 & & (z_0, z_1, z_2) & \longmapsto & (z_0^2, z_1^2 + \sqrt{-1}z_2^2, z_1^2 - \sqrt{-1}z_2^2) & &
\end{array}$$

Note that μ is a finite map with degree 4 ramifying along (w_0), $(w_1 \pm w_2)$. The tangents (w_2) and (w_1) of the conic $(w_0^2 + w_1 w_2)$ at the intersection points $(0,1,0)$ and $(0,0,1)$, respectively, with the line (w_0), pass through the intersection point $(1,0,0)$ of two lines $(w_1 \pm w_2)$. The tangents $(\mp 2w_0 + w_1 - w_2)$ (resp. $(\mp 2w_0 + \sqrt{-1}w_1 + \sqrt{-1}w_2)$) of the conic at the intersection points $(\pm 1, 1, -1)$ (resp. $(\pm\sqrt{-1}, 1, 1)$) with $(w_1 + w_2)$ (resp. $(w_1 - w_2)$), pass through the intersection point $(0,1,1)$ (resp. $(0,1,-1)$) of (w_0) and $(w_1 - w_2)$ (resp. $(w_1 + w_2)$). Hence the pull-back under μ of each of these six tangents consists of two lines tangent to $(z_0^4 + z_1^4 + z_2^4)$ of order 4 at a point. Thus we see that ϕ has six singular fibers of type I_4. Besides twelve tangents $((z_i + \omega^k z_j)$ $(0 \le i < j \le 2$, $k = 1, 3, 5$ or 7, $\omega = \exp(2\pi\sqrt{-1}/8)))$ of $(z_0^4 + z_1^4 + z_2^4)$ which appear in the above, there exist sixteen bitangents $(z_0 + \sqrt{-1}^k z_1 + \sqrt{-1}^l z_2)$ $(0 \le k, l \le 3)$, which are tangent to $(z_0^4 + z_1^4 + z_2^4)$ at two points. Hence by Proposition 14, any fibration ψ on Y satisfying $\psi \circ \sigma^2 = \psi$ except ϕ, $\phi \circ \sigma$ has at least one singular fiber of type I_2 or I_3.

Example 6. Let μ_1 and μ_2 be as in Example 3. Then we see as in the above example that the fibrations induced from μ_1 have three singular fibers of type I_2^*, and those induced from μ_2 have two singular fibers of type I_2^* and four singular fibers of type I_2. (The pull-backs $(4z_1z_2 \pm 4z_0^2 \mp z_1^2 \mp z_2^2)$ under μ_2 of the tangents $(4w_0 \pm 4w_1 \mp w_2)$ of $(w_0^2 + w_1 w_2)$ at $(2, \mp 1, \pm 4)$ are smooth conics tangent to C of order 4 at two points.)

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